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A REFINED FIRST-ORDER SHEAR-DEFORMATION THEORY AND ITS JUSTIFICATION BY PLANE-STRAIN BENDING PROBLEM OF LAMINATED PLATES

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Abstract—A refined first-order shear-deformation theory is proposed and used to solve the plane-strain bending problem of both homogeneous plates and symmetric cross-ply laminated plates. In Reissner–Mindlin's traditional first-order shear-deformation theory (FSDT), the displacement field assumptions include a linear inplane displacement component and a constant transverse deflection through the thickness. These assumptions are retained in the present refined theory. However, the associated transverse shear strain derived from these displacement assumptions, which is still independent of the thickness coordinate, is endowed with new meaning—the stress-weighted average shear strain through the thickness. The variable distribution of transverse shear strain is assumed in such a way that it agrees with the shear stress distribution derived from the integration of equilibrium equation. This paper introduces the effective transverse shear stiffness of plates by assuming that the normalized distribution of through-the-thickness transverse shear stress remains unchanged regardless of geometrical configuration (span-to-thickness ratio) for plane-strain bending problem, which is justified by the exact elasticity solution. Without losing the simplicity of the displacement field assumptions of Reissner–Mindlin's FSDT, the present refined first-order theory not only shows improvement on predicting deflections but also accounts for a variable transverse shear strain distribution through the thickness. In addition, all the boundary conditions, equilibrium equations, and constitutive relations are satisfied pointwise. Comparisons of deflection, transverse shear strain, and transverse shear stress obtained using the present theory are made with the exact results given by Pagano.

INTRODUCTION

While composite materials offer advantages over conventional materials, they also pose challenging technical problems in predicting their structural response. One inherent feature of composite laminates is that the transverse shear modulus is lower than the inplane moduli, and as a result, the influence of transverse shear deformations becomes significant as the plate thickness increases. Classical plate theory predicts the response of thin isotropic plates with reasonable accuracy, yet it usually fails to yield similar accuracy for composite plates of similar configuration.

The first-order shear-deformation theory (FSDT) proposed by Reissner (1945) and Mindlin (1951) assumes that the inplane displacement field is linear and the transverse deflection field is constant through the thickness. It results in a fairly accurate global response for isotropic materials when used with an appropriate shear correction factor, even though a parabolic transverse shear strain distribution through the thickness is not described. Yang *et al.* (1966) extended this theory to laminated plates, followed by many variants of the first-order theory. Reissner (1985), Noor and Burton (1989), Reddy (1990) have reviewed these developments. Extension of FSDT to laminated anisotropic plates has not been as successful as it has been for isotropic plates—particularly for the recovery of the interlaminar stress state without integrating the equilibrium equations. It is also difficult to determine properly the shear correction factor of laminates, upon which the accuracy of the prediction of FSDT is strongly dependent.

Many theories have been developed to overcome the deficiency of FSDT—a constant or uniform transverse shear strain distribution through the thickness. Whitney and Sun (1973) proposed a second-order theory, which allows a linear variation of transverse shear strain through the thickness. Various third-order theories that lead to a parabolic distribution of transverse shear strain through the thickness have also been developed [e.g. Lo *et al.* (1977); Murthy (1981); Reddy (1984)]. For both homogeneous and laminated plates, the transverse shear stresses are continuous through the thickness of the plates according to elasticity theory. However, the transverse shear strains in laminated plates, unlike those in homogeneous plates, generally exhibit discontinuities at layer interfaces due to dissimilar properties of neighboring layers. In that sense, most higher-order theories are inadequate because they either lead to or are based on continuous transverse shear strain distributions through the thickness of the laminated plates. As a result, a postprocessing procedure is usually required to recover the actual interlaminar stress state by integrating the equilibrium equations. Babuska *et al.* (1992) presented a hierarchic modeling approach which can be expanded depending on the goals of computation and regions of interest. Unlike most higher-order theories, the displacement field in their hierarchic theory is assumed to have an exponential nature with the precise form determined by satisfying the equilibrium equations in the transverse direction to the required degree of accuracy. Layer-wise theories [e.g. Reddy *et al.* (1989); Toledano and Murakami (1987)] have been developed which usually assume separate displacement field expansions within each layer and thereby provide a more kinematically correct representation of the strain field in each discrete layer of the laminate and also allow accurate ply-level stresses to be determined. However, since the number of independent field variables is directly proportional to the number of plies in a laminate, the application of a layer-wise theory can be computationally prohibitive when used to model realistic structures and thus is usually applied only in local domains of interest. Finite element formulations based on layer-wise theories are used only in a global-local fashion—the global domain is modeled by FSDT while the local domain of particular significance is modeled using a layer-wise theory.

The first-order shear-deformation theory, from an engineering point of view, is still the most attractive approach due to its simplicity and low computational cost. It is well recognized that while FSDT is adequate for global structural behavior (e.g. transverse deflections, fundamental vibration frequencies, critical buckling loads, force and moment resultants), it is not adequate for accurate prediction of local response parameters, such as the interlaminar stress distributions wherein the transverse shear strains derived from the displacement field assumptions are evenly distributed or uniform through the thickness.

In this paper, a refined first-order deformation theory is presented and used to solve the plane-strain bending problem of plates as a justification. The present refined theory retains the basic displacement assumptions of Reissner–Mindlin's traditional FSDT (i.e. inplane displacement varies linearly and the transverse displacement is constant through the thickness). The transverse shear strain derived from these displacement assumptions remains constant or uniform through the thickness and is referred to herein as the *nominal uniform transverse shear strain*. This term is shown to be the transverse-shear-stress-weighted-average transverse shear strain through the thickness based on the equivalent shear strain energy. The actual transverse shear strain distribution in the present refined theory is no longer assumed to be uniform, instead, it exhibits a variable distribution which correlates with the actual transverse shear stress distribution in a constitutive relationship sense. For an isotropic plate, the transverse shear stress distribution through the thickness takes a parabolic form when obtained by integrating the equilibrium equation. Hence the transverse shear strain should exhibit a parabolic form since it may be recovered from the shear stresses through the use of the constitutive relations. However, it is more common to obtain the transverse shear strain by using the strain-displacement relation which gives a uniform distribution through the thickness, and hence a uniform transverse shear stress. This contradiction within the traditional FSDT which has been plaguing this theory ever since it came into being is removed in the present refined theory. For plane-bending problem of plates, the normalized distribution shape of transverse shear stress is assumed to be independent of the geometric configuration of the plate and external loading type or distribution. With such an assumption, the shear strain distribution is modelled and related to the nominal uniform transverse shear strain based on the equivalent shear strain energy. A shear correction factor is not involved. The novel concept of *effective transverse shear*

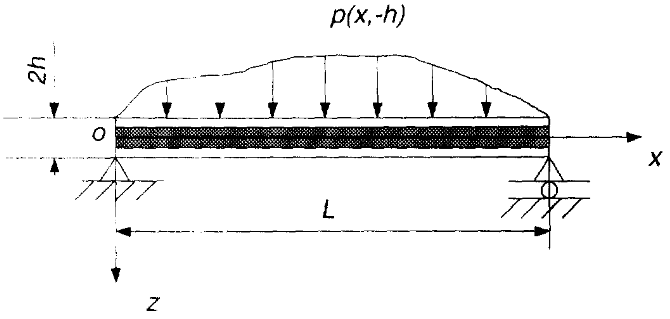

Fig. 1. Plate configuration.

stiffness, is introduced as the counterpart to bending stiffness. As such, a new vitality is offered to FSDT without losing its simplicity. Furthermore, the refined theory not only accounts for the correct through-the-thickness transverse shear strain distribution which may be continuous or discontinuous but also satisfies pointwise all the equilibrium equations, constitutive relations and boundary conditions for plates made of either isotropic or anisotropic materials.

THEORETICAL ANALYSIS

This paper focuses on the plane-bending problem of semi-infinite simply supported, symmetric cross-ply laminated plates. As illustrated in Fig. 1, the geometry of the plate has a total thickness of $2h$ in Z direction, a span of length L in X direction and an infinite length in Y direction. The plate is relatively thin for which the displacement assumptions of FSDT generally hold (further discussion is included later). A lateral loading $p(x)$ in XOZ plane is applied. Discussions of isotropic and homogeneous plates are also included as special cases of laminated plates.

A nondimensional parameter ζ is introduced for the thickness coordinate and is given by

$$\zeta = \frac{z}{h}, \quad \zeta \in [-1, 1]. \tag{1}$$

Reissner–Mindlin's traditional FSDT requires the following form for the displacement field with the midplane displacement in the X direction being zero

$$\begin{aligned} u_x(x, z) &= z\theta(x) \\ u_z(x, z) &= w(x) \end{aligned} \tag{2}$$

where u_x, u_z are displacements in the X, Z directions, respectively, and θ is the rotation angle about the Y axis (normal to the page).

The strain fields derived from these displacement assumptions are expressed as

$$\begin{aligned} \varepsilon_x &= \frac{\partial u_x}{\partial x} = z\theta_{,x} \\ \gamma_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \theta + w_{,x} \end{aligned} \tag{3}$$

where a subscript comma denotes differentiation with respect to that independent variable ensued.

The linear form of ε_x in eqn (3) is known to express sufficiently the actual distribution of the inplane normal strain. However, the transverse shear strain so obtained is uniform or constant through the thickness and does not adequately represent the actual transverse shear strain distribution. The transverse shear strain given in eqn (3), denoted by $\bar{\gamma}_{xz}$ for distinction, is referred herein to as the *nominal uniform transverse shear strain*. The nominal uniform transverse shear strain $\bar{\gamma}_{xz}$ and displacement field variables θ and w are hence related by eqn (3).

For this class of problem, the generalized Hooke's law can be written for the kth layer of the laminate as

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix}^{(k)} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & c_{55} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} \tag{4}$$

where the strain components are the actual ones. The actual transverse shear strain herein is denoted by γ_{xz} and should be distinguished from the nominal uniform transverse shear strain $\bar{\gamma}_{xz}$ given in eqn (3).

The transverse normal stress σ_z, which is rather small compared to the inplane stress, is neglected. Further comments related to this assumption will be made in the discussion of results. As a consequence, the inplane normal stress for the kth layer can be obtained from eqn (4) as

$$\sigma_x^{(k)} = \left[c_{11}^{(k)} - \frac{(c_{13}^{(k)})^2}{c_{33}^{(k)}} \right] \varepsilon_x = Q_{11}^{(k)} \varepsilon_x. \tag{5}$$

In order to determine the transverse shear stress, one of the equilibrium equations with the body force neglected, namely,

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{6}$$

is used. Eqn (6) with eqns (3) and (5), plus the boundary condition that the outer surface of the plate have no transverse shear stress yields

$$\tau_{xz}(x, z) = \tau_{xz}(x, h) + \int_z^h \frac{\partial \sigma_x}{\partial z} \, \mathrm{d}z = \int_z^h z\theta_{,xx} Q_{11}^{(k)} \, \mathrm{d}z. \tag{7}$$

Or in terms of the nondimensional parameter ζ in the thickness direction,

$$\tau_{xz}(x, \zeta) = h^2 \theta_{,xx} \int_\zeta^1 \zeta Q_{11}^{(k)} \, \mathrm{d}\zeta. \tag{8}$$

Introducing a distribution shape function of transverse shear stress as

$$H_z(\zeta) = \frac{4h}{3V(x)} \tau_{xz}(x, \zeta) = \frac{\displaystyle\int_\zeta^1 \eta Q_{11}^{(k)} \, \mathrm{d}\eta}{\displaystyle\frac{3}{4} \int_{-1}^1 \left(\int_\zeta^1 \eta Q_{11}^{(k)} \, \mathrm{d}\eta \right) \mathrm{d}\zeta} \tag{9}$$

where $V(x)$ is the shear stress resultant given by

$$V(x) = \int_{-1}^{1} \tau_{xz}(x,\zeta) h \, d\zeta = h^3 \theta_{,xx} \int_{-1}^{1} \left(\int_{\zeta}^{1} \eta Q_{11}^{(k)} \, d\eta \right) d\zeta. \tag{10}$$

The coefficient 4/3 is employed in order to simplify the distribution function, which will be evident soon. The integral in the denominator of eqn (9) evaluates to some constant and the integral in the numerator evaluates to a continuous piecewise quadratic function—it is quadratic within each layer and continuous at the layer interfaces. Thus the distribution shape of transverse shear stress, expressed by its distribution shape function of variable ζ only, can be described by a continuous piecewise quadratic function. This distribution shape will be illustrated in the Numerical Results and Comparison Section. For the isotropic or homogeneous case, where $Q_{11}^{(k)}$ is constant and simplified to Q_{11}, the transverse shear stress distribution shape function has the well-known parabolic form through the thickness

$$H_\tau(\zeta) = 1 - \zeta^2 \tag{11}$$

which agrees with the surface traction-free boundary conditions.

The distribution shape function of transverse shear strain, $H_\gamma(\zeta)$, can be determined using the constitutive relation and the shear stress distribution shape function. That is,

$$H_\gamma(\zeta) = \frac{\dfrac{H_\tau(\zeta)}{c_{55}^{(k)}}}{\dfrac{3}{4}\displaystyle\int_{-1}^{1} \frac{H_\tau(\zeta)}{c_{55}^{(k)}} \, d\zeta} = \frac{H_\tau(\zeta)}{S c_{55}^{(k)}} \tag{12}$$

where S is the *effective transverse shear compliance* of the plate, or more appropriately the transverse-shear-stress-weighted-average transverse shear compliance of the plate and defined as

$$S = \frac{\displaystyle\int_{-1}^{1} \frac{H_\tau(\zeta)}{c_{55}^{(k)}} \, d\zeta}{\displaystyle\int_{-1}^{1} H_\tau(\zeta) \, d\zeta} = \frac{3}{4} \int_{-1}^{1} \frac{H_\tau(\zeta)}{c_{55}^{(k)}} \, d\zeta. \tag{13}$$

For isotropic case, the effective transverse shear compliance of the plate can be shown to be the reciprocal of the shear modulus of the material.

The distribution shape function of transverse shear strain, $H_\gamma(\zeta)$, is also piecewise quadratic through the thickness. However, it generally is discontinuous at dissimilar layer interfaces due to different values of $c_{55}^{(k)}$ for laminated plates. For isotropic plates, the distribution shape function of transverse shear strain also simplifies to the expression

$$H_\gamma(\zeta) = 1 - \zeta^2. \tag{14}$$

Hence the through-the-thickness distributions of transverse shear stress and transverse shear strain are consistent. Unlike in the traditional FSDT, no inconsistency with the constitutive relation exists in the present refined theory, since the transverse shear strain takes the same parabolic distribution form as its stress counterpart.

Both distribution shape functions of transverse shear stress and shear strain, $H_\tau(\zeta)$ in eqn (9) and $H_\gamma(\zeta)$ in eqn (12) are normalized in the sense that

$$\int_{-1}^{1} H_\tau(\zeta)\,\mathrm{d}\zeta = \int_{-1}^{1} H_\gamma(\zeta)\,\mathrm{d}\zeta = \int_{-1}^{1} (1-\zeta^2)\,\mathrm{d}\zeta = \frac{4}{3} \tag{15}$$

which accounts for why a factor of 3/4 or 4/3 appears in eqns (9), (12) and (13).

Assume the effective magnitudes of transverse shear stress and shear strain are denoted as τ_{xz}^{em} and γ_{xz}^{em}, respectively, then the actual transverse shear stress and shear strain through the thickness are expressed accordingly as

$$\begin{aligned}\tau_{xz}(x,\zeta) &= \tau_{xz}^{em}(x)H_\tau(\zeta)\\ \gamma_{xz}(x,\zeta) &= \gamma_{xz}^{em}(x)H_\gamma(\zeta).\end{aligned} \tag{16}$$

For isotropic plates where $H_\tau(\zeta)$ and $H_\gamma(\zeta)$ take the special parabolic form as expressed by eqns (11) and (14), τ_{xz}^{m} and γ_{xz}^{em} are their respective magnitudes, or the maximum values. Otherwise, the effective magnitudes are expected to differ from their corresponding peak values.

The total transverse shear energy U_s through the thickness is expressed as

$$U_s(x) = \frac{1}{2}\int_{-1}^{1} \tau_{xz}(x,\zeta)\gamma_{xz}(x,\zeta)h\,\mathrm{d}\zeta. \tag{17}$$

Using the definition given by eqn (16), the transverse shear strain energy takes the form

$$U_s(x) = \tfrac{1}{2}h\tau_{xz}^{em}(x)\gamma_{xz}^{em}(x)\int_{-1}^{1} H_\tau(\zeta)H_\gamma(\zeta)\,\mathrm{d}\zeta. \tag{18}$$

On the other hand, the transverse shear strain energy can also be expressed in terms of the average shear strain. Namely,

$$\bar{U}_s(x) = \tfrac{1}{2}\bar{\gamma}_{xz}(x)\int_{-1}^{1} \tau_{xz}(x,\zeta)h\,\mathrm{d}\zeta = \tfrac{1}{2}h\bar{\gamma}_{xz}(x)\tau_{xz}^{em}(x)\int_{-1}^{1} H_\tau(\zeta)\,\mathrm{d}\zeta. \tag{19}$$

Equating these two transverse shear strain energy expressions ($U_s = \bar{U}_s$) and solving for the nominal uniform transverse shear strain $\bar{\gamma}_{xz}$ give

$$\bar{\gamma}_{xz}(x) = \frac{\int_{-1}^{1} \tau_{xz}(x,\zeta)\gamma_{xz}(x,\zeta)\,\mathrm{d}\zeta}{\int_{-1}^{1} \tau_{xz}(x,\zeta)\,\mathrm{d}\zeta} = \frac{\int_{-1}^{1} H_\tau(\zeta)H_\gamma(\zeta)\,\mathrm{d}\zeta}{\int_{-1}^{1} H_\tau(\zeta)\,\mathrm{d}\zeta}\gamma_{xz}^{em}(x). \tag{20}$$

Alternatively, the ratio of transverse shear strain effective magnitude γ_{xz}^{em} to the nominal uniform shear strain $\bar{\gamma}_{xz}$ is

$$f_z = \frac{\gamma_{xz}^{em}(x)}{\bar{\gamma}_{xz}(x)} = \frac{\int_{-1}^{1} H_\tau(\zeta)\,\mathrm{d}\zeta}{\int_{-1}^{1} H_\tau(\zeta)H_\gamma(\zeta)\,\mathrm{d}\zeta}. \tag{21}$$

For isotropic plates, the value of f_z can be shown to be 5/4.

The final expressions for the actual transverse shear strain and shear stress are, respectively

$$\gamma_{xz}(x,\zeta) = \gamma_{xz}^{cm}(x)H_j(\zeta) = f_j\bar{\gamma}_{xz}(x)H_j(\zeta) \tag{22}$$

and

$$\tau_{xz}(x,\zeta) = \tau_{xz}^{cm}H_j(\zeta) = \frac{f_j\bar{\gamma}_{xz}(x)}{S}H_j(\zeta). \tag{23}$$

It follows that the transverse shear stress given by eqn (23) is continuous and piecewise quadratic through the thickness since $H_j(\zeta)$ is continuous. Likewise the transverse shear strain given by eqn (22) is only piecewise quadratic because of the discontinuity of $H_j(\zeta)$ at dissimilar layer interfaces. Such a result agrees with elasticity theory.

Using the actual transverse shear strain and stress expressions given by eqns (22) and (23), respectively, the actual shear strain energy U_s can be also expressed as

$$\begin{aligned} U_s(x) &= \frac{1}{2}\int_{-1}^{1}\tau_{xz}(x,\zeta)\gamma_{xz}(x,\zeta)h\,\mathrm{d}\zeta \\ &= \frac{1}{2}\left[hf_j^2\int_{-1}^{1}c_{55}^{(k)}H_j^2(\zeta)\,\mathrm{d}\zeta\right]\bar{\gamma}_{xz}^2(x) = \tfrac{1}{2}F\bar{\gamma}_{xz}^2(x) \end{aligned} \tag{24}$$

which depends only on the nominal uniform transverse shear strain $\bar{\gamma}_{xz}$ and a new parameter F that is defined as

$$F = hf_j^2\int_{-1}^{1}c_{55}^{(k)}H_j^2(\zeta)\,\mathrm{d}\zeta = \int_{-1}^{1}c_{55}^{(k)}[f_jH_j(\zeta)]^2h\,\mathrm{d}\zeta. \tag{25}$$

The parameter F represents the *effective transverse shear stiffness* — a counterpart to bending stiffness. It is interesting to compare eqns (24) and (25) with the expression of bending strain energy

$$U_b(x) = \tfrac{1}{2}D_{11}\kappa_x^2(x) \tag{26}$$

and the definition of bending stiffness

$$D_{11} = \int_{-1}^{1}Q_{11}^{(k)}[h\zeta]^2h\,\mathrm{d}\zeta \tag{27}$$

where $\kappa_x(x)$ is the inplane strain curvature. The expression of transverse shear strain energy, eqn (24), is analogous to that of bending strain energy, eqn (26). The definition of effective transverse shear stiffness, eqn (25), is likewise analogous to that of bending stiffness, eqn (27). Both $c_{55}^{(k)}$ in eqn (25) and $Q_{11}^{(k)}$ in eqn (27) are elastic stiffness coefficients. The term $f_jH_j(\zeta)$ in eqn (25) can represent the distribution shape of transverse shear strain [i.e. $\gamma_{xz}(x,\zeta) = f_jH_j(\zeta)\bar{\gamma}_{xz}(x)$] whereas the term $h\zeta$ in eqn (27) represents the linear distribution form of the inplane normal strain due to pure bending [i.e. $\varepsilon_x^{b}(x,\zeta) = h\zeta\kappa_x(x)$].

Again for isotropic material where

$$f_j = 5/4, \quad H_j(\zeta) = 1-\zeta^2, \quad c_{55}^{(k)} = G = \frac{E}{2(1+\nu)} \tag{28}$$

the effective transverse shear stiffness F is given by

$$F = hf^2 \int_{-1}^{1} c_{55}^{(k)} H^2(\zeta)\,\mathrm{d}\zeta = \tfrac{5}{6}(2h)G \tag{29}$$

and the transverse shear strain energy is accordingly given by

$$U_s(x) = \tfrac{1}{2}F\bar{\gamma}_{xz}^2(x) = \tfrac{1}{2}\times(\tfrac{5}{6})(2h)G\bar{\gamma}_{xz}^2(x). \tag{30}$$

This result coincides with the widely accepted transverse shear strain energy expression with a *shear correction factor* equal to 5/6.

The following summary is made in retrospective at this point. The nominal uniform transverse shear strain $\bar{\gamma}_{xz}$ derived based on FSDT displacement assumptions is actually a weighted-average transverse shear strain through the thickness, and the weighting function is the corresponding transverse shear stress. Alternatively, the nominal uniform transverse shear strain, expressed by eqn (20), has been shown herein to be the stress-weighted average shear strain through the thickness based on the equivalent shear strain energy. Actual transverse shear stress and shear strain are both assumed to vary through the thickness and are consistently related by the constitutive relationship. Their distributions can be expressed by their respective effective magnitudes and distribution shape functions as given by eqns (22) and (23). Their effective magnitudes are related by the effective transverse shear compliance of the plate.

Specifically for an isotropic material, the shear stress distribution shape function and weighting function is

$$H(\zeta) = 1-\zeta^2. \tag{31}$$

As such, the nominal uniform transverse shear strain becomes

$$\bar{\gamma}_{xz}(x) = \frac{\int_{-1}^{1} H(\zeta)\gamma_{xz}(x,\zeta)\,\mathrm{d}\zeta}{\int_{-1}^{1} H(\zeta)\,\mathrm{d}\zeta} = \frac{\int_{-1}^{1} (1-\zeta^2)^2\gamma_{xz}^{em}(x)\,\mathrm{d}\zeta}{\int_{-1}^{1} (1-\zeta^2)\,\mathrm{d}\zeta} = \tfrac{4}{5}\gamma_{xz}^{em}(x). \tag{32}$$

In some sense, the present approach is analogous to Tessler and Saether's (1991) treatment of transverse deflection for isotropic materials, which takes the form

$$u_z(x,\zeta) = w(x) + \zeta w_1(x) + (\zeta^2 - \tfrac{1}{5})w_2(x) \tag{33}$$

wherein $w(x)$ is the weighted displacement average [see Reissner (1950)]

$$w(x) = \frac{\int_{-1}^{1} (1-\zeta^2)u_z(x,\zeta)\,\mathrm{d}\zeta}{\int_{-1}^{1} (1-\zeta^2)\,\mathrm{d}\zeta} = \frac{3}{4}\int_{-1}^{1} (1-\zeta^2)u_z(x,\zeta)\,\mathrm{d}\zeta. \tag{34}$$

For the purpose of demonstrating the validity of the present refined theory, we now come to a special plane-strain bending problem—cylindrical bending problem of orthotropic plates for which Pagano (1969) developed an exact elasticity solution. Assuming the lateral load is $p(x)$ and neglecting transverse normal strain energy, the total potential energy per unit length in the Y direction is expressed as

$$\Pi = \int_0^l (U_b + U_s - pw)\,\mathrm{d}x$$

$$= \int_0^l \left[\frac{D}{2}(\theta_{,x})^2 + \frac{F}{2}(w_{,x} + \theta)^2 - pw \right] \mathrm{d}x \tag{35}$$

where U_b is the bending energy through the thickness and the bending stiffness D (or D_{11}) is

$$D = \int_{-h}^{h} z^2 Q_{11}^{(k)}\,\mathrm{d}z. \tag{36}$$

The Euler–Lagrange equations from the total potential energy functional given by eqn (35) are

$$\begin{aligned} -D\theta_{,xx} + F(w_{,x} + \theta) &= 0 \\ F(w_{,xx} + \theta_{,x}) &= -p \end{aligned} \tag{37}$$

which are similar to the expressions of Timoshenko's beam theory except for the newly introduced effective transverse shear stiffness F.

The Euler–Lagrange equations may be uncoupled to give

$$D\frac{\mathrm{d}^3\theta}{\mathrm{d}x^3} = -p$$

$$D\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = p - \frac{D}{F}\left(\frac{\mathrm{d}^2 p}{\mathrm{d}x^2}\right). \tag{38}$$

Clearly the transverse shear effect will disappear when $p(x)$ is either uniform or linear function of x. In such cases, the plates have a tendency to only translate or rotate and the overall transverse shear strain through the thickness section will be zero. The corresponding deflection and rotation expressions may be derived using eqns (37) and (38) plus appropriate boundary condition. As the distribution shape functions of transverse shear stress and shear strain can be obtained from the plate material configuration, the actual through-the-thickness transverse shear stress and shear strain can be readily determined.

NUMERICAL RESULTS AND COMPARISON

The exact solution for cylindrical bending problem of cross-ply laminated plates proposed by Pagano (1969) is used as the benchmark solution for the present refined theory.

Comparisons of deflection results with the exact solution w_e are made with:

(1) Solution by present refined first-order shear deformation theory employing effective transverse shear stiffness, denoted by w_p.

(2) Classical solution with shear deformation neglected, or, assuming the transverse shear stiffness to be infinity, denoted by w_c.

(3) Solution by Reissner–Mindlin's traditional FSDT using average transverse shear modulus and assuming 5/6 as the shear correction factor, denoted by w_m.

(4) Solution by Reissner–Mindlin's traditional FSDT using average transverse shear modulus and assuming the shear correction factor to be 1, denoted by w_{m1}.

The ply material properties corresponding to a typical carbon epoxy materials are taken as

Table 1. Laminate definitions

No. of layers	Stacking sequence	D (in.-lb)	F (in.-lb)	F_a (in.-lb)
1	[0]	0.261	2083	2083
4	$[90/0]_s$	2.673	5051	5833
16	$[90/0]_{4s}$	459.8	18620	23330
16	$[90_3/0_3/90/0]_s$	279.4	15930	23330

$$E_L = 25 \times 10^6 \text{ psi} \quad E_T = 10^6 \text{ psi}$$
$$G_{LT} = 0.5 \times 10^6 \text{ psi} \quad G_{TT} = 0.2 \times 10^6 \text{ psi}$$
$$\nu_{LT} = \nu_{TT} = 0.25 \tag{39}$$

where L and T denote the longitudinal and transverse ply material directions, respectively. Using these properties, four laminates are considered. For each case, the layer properties are given in eqn (39), and each layer is assumed to have the same thickness—0.005 in, while the plate span varies so that the ratio of span-to-thickness, $R = L/2h$, takes on values 4, 10, 20 and 50, respectively.

In Table 1, D is defined by eqn (36), F by eqn (25) and F_a is the average effective shear stiffness defined by

$$F_a = \frac{5}{6}\int_{-h}^{h} C_{55}^{(k)}\,\mathrm{d}z \tag{40}$$

which agrees with Reissner–Mindlin's traditional FSDT if a shear correction factor of 5/6 is used. *Average* means that the stacking sequence is not taken into account, while *effective* means that the conventional shear correction factor 5/6 is used.

The single-layer plate is obviously the representation of homogeneous case which also includes the isotropic plate as a specific one. The two 16-ply plates have the same value for F_a in Table 1; however, they have significantly different values for the effective transverse shear stiffness F, since the latter parameter takes stacking consequence into account, just like the bending stiffness.

For cylindrical bending problem, the lateral load is assumed to have the form

$$p(x, h) = \tfrac{1}{2}p_0 \sin\frac{\pi x}{L}$$
$$p(x, -h) = -\tfrac{1}{2}p_0 \sin\frac{\pi x}{L} \tag{41}$$

and the corresponding boundary conditions are

$$w = 0, \quad M_x = 0 \quad \text{at } x = 0, \; L. \tag{42}$$

The analytic solution is readily obtained and given as

$$w(x) = \frac{p_0}{D}\left(\frac{L}{\pi}\right)^4 \sin\left(\frac{\pi x}{L}\right)\left[1 + \frac{D}{F}\left(\frac{\pi}{L}\right)^2\right] \tag{43}$$

and

$$\theta(x) = \frac{p_0}{D}\left(\frac{L}{\pi}\right)\cos\frac{\pi x}{L}. \tag{44}$$

The nominal uniform transverse shear strain is then

$$\bar{\gamma}_x(x) = w_{,x} + \theta = \frac{p_0}{F}\left(\frac{L}{\pi}\right)\cos\frac{\pi x}{L}. \tag{45}$$

The actual distributions of transverse shear strain and shear stress can be expressed as, respectively

$$\gamma_{xz}(x,\zeta) = f\bar{\gamma}_x(x)H_1(\zeta) = f\frac{p_0}{F}\left(\frac{L}{\pi}\right)\cos\left(\frac{\pi x}{L}\right)H_1(\zeta) \tag{46}$$

and

$$\tau_{xz}(x,\zeta) = c_{55}^{(k)}\gamma_{xz}(x,\zeta) = \frac{f}{S}\frac{p_0}{F}\left(\frac{L}{\pi}\right)\cos\left(\frac{\pi x}{L}\right)H_1(\zeta). \tag{47}$$

The transverse shear stress resultant derived from eqn (47) can be shown to be precisely the same as that obtained from an elasticity analysis.

For the classical solution in which the effect of transverse shear deformation is neglected [i.e. F approaches infinity in eqn (43)] the corresponding expression for the transverse deflection at the midplane of the plate is given by

$$w_c(x) = \frac{p_0}{D}\left(\frac{L}{\pi}\right)^4\sin\frac{\pi x}{L}. \tag{48}$$

If F in eqn (43) is replaced by F_u defined by eqn (40), the result for the traditional FSDT is obtained with the shear correction factor equal to 5/6. That is,

$$w_u(x) = \frac{p_0}{D}\left(\frac{L}{\pi}\right)^4\sin\left(\frac{\pi x}{L}\right)\left[1 + \frac{D}{F_u}\left(\frac{\pi}{L}\right)^2\right] \tag{49}$$

In case that the shear correction factor is assumed to be 1 instead of 5/6, the corresponding result is

$$w_{u1}(x) = \frac{p_0}{D}\left(\frac{L}{\pi}\right)^4\sin\left(\frac{\pi x}{L}\right)\left[1 + \frac{D}{\frac{6}{5}F_u}\left(\frac{\pi}{L}\right)^2\right]. \tag{50}$$

The exact solution for such cylindrical problem involving a cross-ply laminate proposed by Pagano (1969), although not as simple as the above expressions, can be expressed as

$$w(x,z) = \left[\sum_{i=1}^{4} A_i\left(R_{12}^{(i)} - \frac{R_{22}^{(i)}}{m_i}\times\frac{\pi^2}{L^2}\right)\exp(m_i z)\right]\sin\frac{\pi x}{L} \tag{51}$$

and the exact value of the midplane deflection (at $ = 0$) is

Table 2. Midplane center transverse deflection comparison in percent error for the single [0] layer

R	$\bar{w}_p$	$\bar{w}_c$	$\bar{w}_m$	$\bar{w}_{m1}$
4	3.19	-75.78	3.19	−9.80
10	0.39	32.84	0.39	−5.15
20	0.07	-10.94	0.07	−1.76
50	0.01	1.93	0.01	−0.31

Table 3. Midplane center transverse deflection comparison in per cent error for the [90/0]$_s$ laminate

R	$\bar{w}_p$	$\bar{w}_c$	$\bar{w}_m$	$\bar{w}_{m1}$
4	2.43	43.60	−3.74	−10.39
10	0.29	-11.30	−1.27	−2.94
20	0.07	-3.10	−0.36	−0.82
50	0.01	-0.51	−0.06	−0.14

Table 4. Midplane center transverse deflection comparison in per cent error for the [90/0]$_{4s}$ laminate

R	$\bar{w}_p$	$\bar{w}_c$	$\bar{w}_m$	$\bar{w}_{m1}$
4	2.01	-69.82	−12.50	−22.06
10	0.28	27.38	−5.32	−8.99
20	0.06	−8.64	−1.70	−2.85
50	0.01	1.49	−0.29	−0.49

Table 5. Midplane center transverse deflection comparison in per cent error for the [90$_2$/0,90/0]$_s$ laminate

R	$\bar{w}_p$	$\bar{w}_c$	$\bar{w}_m$	$\bar{w}_{m1}$
4	4.06	61.32	−16.67	−24.11
10	0.41	20.96	−6.37	−8.80
20	0.08	−6.26	−1.93	−2.65
50	0.01	1.06	−0.33	−0.45

$$w_e(x) = w(x,0) = \left[\sum_{p=1}^{4} A_p\left(R_{12}^{(p)} - \frac{R_{22}^{(p)}}{m_p}\times\frac{\pi^2}{L^2}\right)\right]\sin\frac{\pi x}{L}. \tag{52}$$

The readers are referred to Pagano (1969) for the full meaning of all terms involved in eqns (51) and (52).

All deflection expressions take the same sinusoidal form along the X direction. The percent error in the midplane transverse deflection at the center of the plate for each approach compared with the exact value given by eqn (52) are obtained. Namely,

$$\bar{w}_i = \frac{w_i - w_e}{w_e}\times 100\% \tag{53}$$

where the subscript i represents each approach ($i = p$ for the present refined theory; $i = c$ for classical plate theory; $i = m$ for Reissner–Mindlin's traditional FSDT with a shear correction factor of 5/6; and $i = m1$ for the traditional FSDT with a shear correction factor of 1). Results are presented in Tables 2–5 for each laminate defined in Table 1.

Results for the single orthotropic layer are shown in Table 2. The present refined FSDT, as anticipated, gives transverse deflection results identical to those of Reissner–Mindlin's traditional FSDT with a shear correction factor equal to 5/6. Even for thick plates ($R = 4$), the present theory and the traditional FSDT are very accurate. Results for the 4-layer cross-ply laminate are shown in Table 3. The present theory is the most accurate one of the results presented and is within 3% of the exact transverse deflection for the thick plate ($R = 4$). Results for the two 16-layer cross-ply laminates are given in Tables 4 and 5 and again indicate the improved accuracy for the present refined theory. The advantage of present theory is more pronounced for thick plates where the transverse shear deformation effect becomes more important. The reason is rather clear: the present refined theory employs more accurate transverse shear strain energy expression.

It is assumed in the previous discussions that normalized distribution of through-the-thickness transverse shear stress remains independent of the span-to-thickness ratio. Such an assumption can also be justified by Pagano's exact solution. All four laminates are examined, yielding conclusive results. Two sets of the results are provided in Fig. 2. The short- and long-dashed lines represent the exact results for two values of the span-to-thickness ratio ($R = 4$, 10, respectively), and the solid line, denoted by *Present*, stands for the results predicted by the present refined theory and determined using eqn (9) for both values of R. The location in the X direction is taken as $x = L/4$. However, the location in X direction makes no difference in the transverse shear stress comparison except at the middle span where the transverse stresses vanish identically.

It can be seen that even for thick plates ($R = 4$), the assumed distribution can be taken as a good approximate expression. When the plate becomes moderately thick ($R = 10$), only slight difference exists between the actual distribution and the assumed one. As the plate becomes thinner and thinner (i.e. $R \to \infty$), the distribution will converge to the assumed one. Results for thinner plates (e.g. $R = 20$, 50), which are not depicted for the sake of clarity, fall between the curve for $R = 10$ and the present assumed result. The assumption that the normalized distribution of through-the-thickness transverse shear stress is independent of the span-to-thickness ratio, especially for the plates not too thick ($R > 4$) can be concluded to be reasonable. Even for thick plate ($R = 4$), this assumption also leads to a good acceptable approximation. Since the present refined theory provides precisely exact transverse shear stress result as mentioned previously, Fig. 2 also represents the comparisons between transverse shear stress distributions obtained by present theory and the exact ones by Pagano's solution for $R = 4$, 10, respectively.

The transverse shear strain results using the present refined theory, eqn (46), are also compared with the exact values and the nominal uniform shear strains as illustrated in Figs 3 and 4. Transverse shear strain comparisons for the single-layer case are shown in Fig. 3. Unlike in the traditional FSDT, the actual distributions of the transverse shear strain are not uniform through the thickness, instead they take the same form as those of the transverse shear stress for homogeneous plates (compare Fig. 3 with Fig. 2). Results for one 16-layer

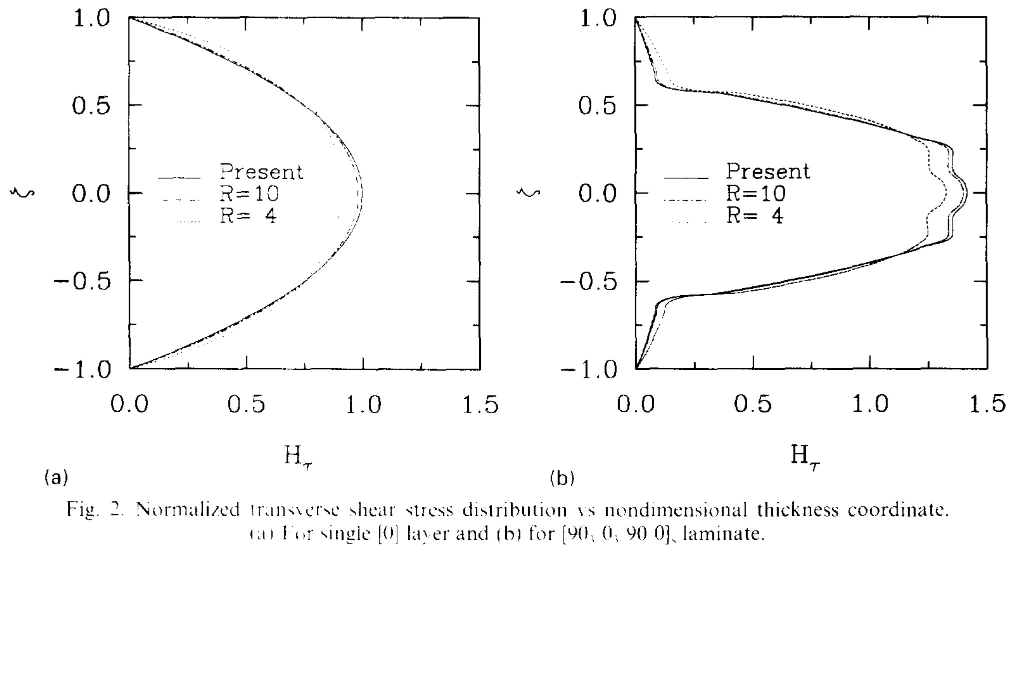

Fig. 2. Normalized transverse shear stress distribution vs nondimensional thickness coordinate. (a) For single [0] layer and (b) for [90, 0, 90 0]$_s$ laminate.

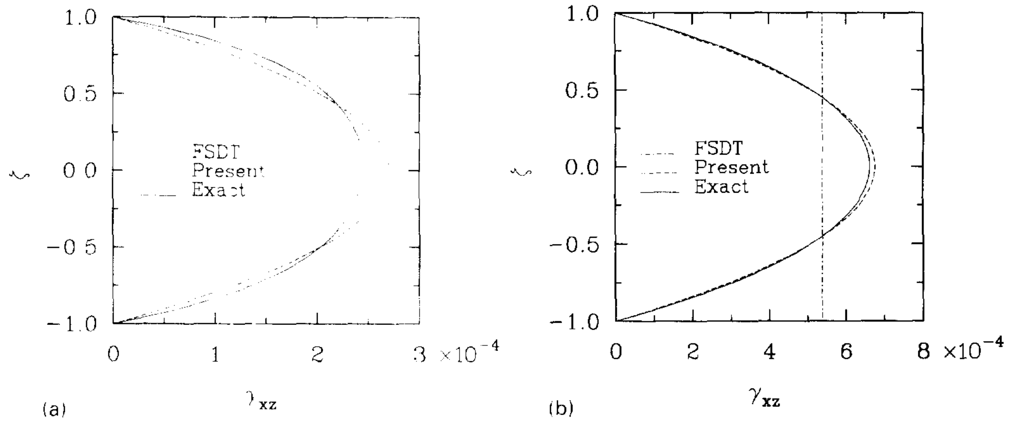

Fig. 3. Transverse shear strain distribution comparison for single [0] layer. (a) $R = 4$ and (b) $R = 10$.

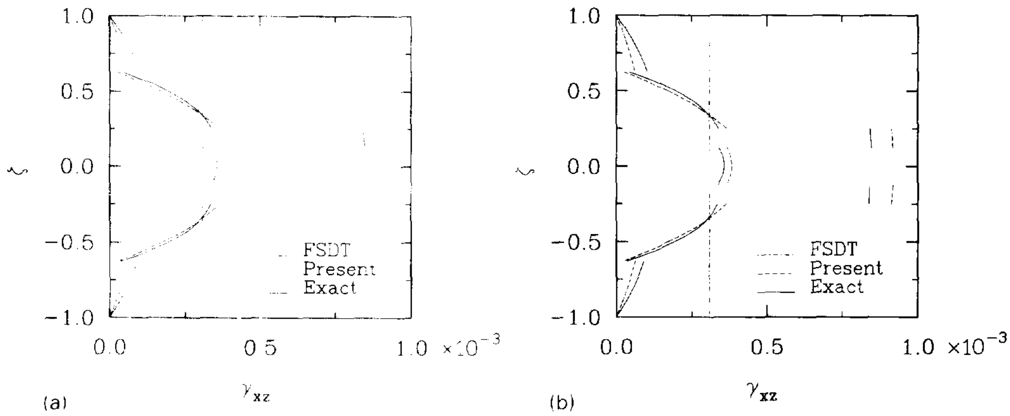

Fig. 4. Transverse shear strain distribution comparison for [90/0/90/0]$_s$ laminate. (a) $R = 4$ and (b) $R = 10$.

laminate are shown in Fig. 4 where the actual distributions of shear strain exhibit enormous diversity. For anisotropic laminated plates, transverse shear strain is neither uniform nor continuous. It shows distinct discontinuity at the dissimilar layer interfaces. Any attempts to assume it is continuous, much less uniform or evenly distributed, are expected to lead to inaccuracies.

The present refined theory, however, rather successfully predicts transverse shear strain distributions, as well as shear stress distributions illustrated in Fig. 2. When the plate is moderately thick ($R = 10$), sufficiently accurate results are obtained by the present refined theory. Results for thin plates ($R = 20, 50$) approach those of the present theory and any differences would not be detected if plotted. Even for thick plates ($R = 4$), the prediction using the present theory is acceptable.

It is widely recognized that for thin plates ($R > 50$ or so), both transverse shear and normal strains can be neglected, and thus the application of classical plate theory generates satisfactory global predictions. When the plates become moderately thick (R is between 10 and 50 or so), the transverse shear effect should be taken into consideration while the transverse normal strain is still negligible. As the plate thickness increases further, the transverse normal strain becomes more important and should be included in the formulation. Usually FSDT and its variants are not appropriate for thick plates ($R = 4$ or so). There are attempts to modify the transverse deflection form of FSDT to include transverse normal strain, for example, assuming quadratic form of transverse deflection in the thickness direction. However, the error introduced by assuming a linear through-the-thickness distribution of inplane displacement which corresponds to a linear through-the-thickness distribution of the inplane stress is more significant than the error introduced by assuming inextensibility in the thickness direction. This conclusion can be easily verified using the exact solution from Pagano but is not included herein.

Even though applied to thick plates in the previous discussion for the sake of comparison, the present refined FSDT should be confined only to moderately thick plates where the transverse shear effect is important while the transverse normal stress can be assumed to be negligible. In the domain of its validity, it has been shown to give accurate adequate global response (transverse deflection) and local response (transverse shear stress and transverse shear strain) prediction for the configuration considered in this paper. In addition, the span-to-thickness ratio should be considered along with the wavelength of the loading function. For a loading function with multiple halfwaves along the plate length, the corresponding *effective* span-to-thickness should be related to the wavelength of the load. The transverse shear effect is still significant for a thin plate if subjected to a very short-wave load.

SUMMARY

In the first part of this paper, the first-order shear-deformation theory is re-examined and it concludes that the nominal uniform transverse shear stress derived directly from the displacement assumptions is actually the stress-weighted average shear strain through the thickness based on the equivalent shear strain energy. The novel explanation enables the present refined FSDT to account for variable distribution of transverse shear strain that Reissner–Mindlin's traditional theory fails to do. The contradiction in the sense of constitutive relationship is therefore removed between the uniform transverse shear strain derived from the displacement assumptions and the variational shear stress obtained from the equilibrium equation.

In the second part, the validity of this refined theory is demonstrated by applying it to solve cylindrical bending problems of plates for which exact results are available. Based on the assumption that normalized transverse shear stress distribution through the thickness remains unchanged, the actual distributions of transverse shear stress and shear strain are modelled which comply with boundary conditions, equilibrium equation and constitutive relationship pointwise. Effective transverse shear stiffness and effective transverse shear compliance of plates are proposed to predict the global response (e.g. deflection, transverse shear stress resultant and nominal uniform transverse shear strain, and likely vibration frequency and buckling load). On the other hand, the local response parameter, such as the interlaminar transverse shear stress and strain distributions, can also be obtained using their respective distribution shape functions.

The underlying explanation that the nominal uniform transverse shear strain derived directly from the displacement assumptions is the stress-weighted average shear strain through thickness enables the present refined theory to succeed not only globally but also locally for the plane-strain bending problems. However, it should be mentioned that for general plate bending problems, the unchanged stress distribution shape assumption may not hold and further research work is required.

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